The vacuum electromagnetic fields and the Schrödinger picture

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Abstract

Several authors have used the *Heisenberg picture* to show that the atomic transitions, the stability of the ground state and the position-momentum commutation relation $[x, p] = i\hbar$, can only be explained by introducing radiation reaction and vacuum electromagnetic fluctuation forces. Here we consider the simple case of a nonrelativistic charged harmonic oscillator, in one dimension, to investigate how to take into account the radiation reaction and vacuum fluctuation forces within the *Schrödinger picture*. We consider the effects of both classical *zero-point* and *thermal* electromagnetic vacuum fields. We show that the zero-point electromagnetic fluctuations are dynamically related to the momentum operator $p = -i\hbar\partial/\partial x$ used in the Schrödinger picture. Consequently, the introduction of the *zero-point* electromagnetic fields in the vector potential $A_x(t)$ used in the Schrödinger equation, generates "double counting", as was shown recently by A.J. Faria et al. (Physics Letters A 305 (2002) 322). We explain, in details, how to avoid the "double counting" by introducing only the radiation reaction and the *thermal* electromagnetic fields into the Schrödinger equation.

Keywords: Foundations of quantum mechanics; Zero-point radiation; Thermal radiation; Stochastic electrodynamics.

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1. Introduction

The question concerning the equivalence between the Schrödinger and the Heisenberg pictures of quantum mechanics was raised a long time ago by P.A.M. Dirac [1]. Recently, A.J. Faria et al. [2] have addressed the problem of the equivalence between the two pictures. In order to clearly explain this problem we start by indicating the importance of the radiation reaction and the vacuum zero-point electromagnetic fields to the understanding of the atomic transitions, and the atomic stability, using the Heisenberg picture and quantum electrodynamics.

Consider a physical system like the hydrogen atom. Its Hamiltonian is

$$H_S = \frac{\vec{p}^{\,2}}{2m} - \frac{e^2}{r} \ , \tag{1}$$

and the atomic states are such that

$$H_S|\mathrm{vac}, a\rangle = \epsilon_a|\mathrm{vac}, a\rangle \ , \tag{2}$$

where ϵ_a is the energy of the atom and $|\mathrm{vac}, a\rangle \equiv |\mathrm{vac}\rangle|a\rangle$ denotes the state in which the atom is in the stationary state $|a\rangle$, and the field is in its vacuum state $|\mathrm{vac}\rangle$ of no photons. Considering the above system, Dalibard, Dupont-Roc and Cohen-Tannoudji [3] have discussed the role of the *vacuum zero-point fluctuations and the radiation reaction* forces, *with the identification of their respective contributions*, in the domain of the atomic transitions with emission of electromagnetic radiation. Considering the conceptual importance of this paper we summarize their main conclusion.

Using a perturbative calculation based on the *Heisenberg picture*, Dalibard et al. concluded that the variation with time of the energy of the system is such that

$$\begin{aligned}\langle \mathrm{vac}, a|\frac{dH_S}{d\,t}|\mathrm{vac}, a\rangle = -\frac{2}{3}\frac{e^2}{c^3}\langle a|(\ddot{\vec{r}})^2|a\rangle + \\ +\frac{2}{3}\frac{e^2}{c^3}\left[\sum_{b\,(\epsilon_b>\epsilon_a)}\langle a|\ddot{\vec{r}}|b\rangle\cdot\langle b|\ddot{\vec{r}}|a\rangle - \sum_{b\,(\epsilon_b<\epsilon_a)}\langle a|\ddot{\vec{r}}|b\rangle\cdot\langle b|\ddot{\vec{r}}|a\rangle\right] \ .\end{aligned} \tag{3}$$

The first term in (3) is the contribution of radiation reaction whereas the second, and the third terms, are the contributions of the vacuum fluctuation forces. It is straightforward to show that (3) can be written as

$$\langle \mathrm{vac}, a|\frac{dH_S}{d\,t}|\mathrm{vac}, a\rangle = -\frac{4}{3}\frac{e^2}{c^3}\sum_{b(\epsilon_b<\epsilon_a)}\langle a|\ddot{\vec{r}}|b\rangle\cdot\langle b|\ddot{\vec{r}}|a\rangle \ . \tag{4}$$

We note that, if self reaction was alone (see the first term in (3)), the atomic ground state would not be stable, since the square of the acceleration has a non zero average value in such a state. Moreover, such a result is extremely simple and exactly coincides with what is found in classical radiation theory [3]. The complete result (see equation (4)), which includes the vacuum forces, is even more satisfactory because the electron in the vacuum can only lose energy by cascading downwards to lower energy levels. The ground state cannot be stable in the absence of vacuum fluctuations which exactly balance the energy loss due to self reaction [4]. In other words, if self reaction was alone, the ground state would collapse and the atomic commutation relation $[x, p] = i\hbar$ would not hold [3]. As stated in reference [3], "all self reaction effects, which are independent of $\hbar$, are strictly identical to those derived from classical radiation theory. All zero-point vacuum fluctuation effects, which are proportional to $\hbar$ can be interpreted by considering the vibration of the electron induced by a random field having a spectral power density equal to $\hbar\omega/2$ per mode". Therefore, in several situations, the zero-point and thermal vacuum fields can be successfully replaced by *classical* random fields [4, 5, 6, 7, 8], so that the electric and magnetic fields can be considered as fluctuating sources of energy.

In order to clarify the features of the interaction between the atom and the vacuum fluctuating fields, we study the statistical properties of a charged harmonic oscillator interacting with vacuum fields, using the *Schrödinger picture*. This is discussed in section 2. We consider separately the effects of each kind of fluctuating field. The effects of the *zero-point* radiation and the radiation reaction are analized within the subsection 2a. The effects of the *thermal* radiation and the radiation reaction are studied within the subsection 2b. Conclusions are presented in the section 3.

2. Charged harmonic oscillator according to the Schrödinger picture

For clarity reason and in order to simplify the calculations, the classical vacuum electric fields to be considered here are the random *zero-point* and *thermal* electric fields of Stochastic Electrodynamics (SED). An excellent review of SED is given in the book by de la Peña and Cetto [8].

We shall assume that the motion of the charged oscillator is non relativistic ($mc^2 >> \hbar\omega_0$) so that the dipole approximation will be used [6]. We shall see that this approximation is consistent with the calculations presented in the subsection 2.1 and 2.2. Following the notation of Boyer [6], the x com-

ponent of the zero-point electric field, acting on the bounded charge moving close to the origin of the coordinate system, is

$$E_x(\vec{r},t)=\sum_{\lambda=1}^{2}\int d^3k\epsilon_x(\vec{k},\lambda)\frac{\sqrt{\hbar\omega/2}}{2\pi}\left[e^{i\theta(\vec{k},\lambda)}e^{-i\omega t}e^{i\vec{k}\cdot\vec{r}}+c.c.\right] . \tag{5}$$

In the long wavelength approximation, one can write this expression as a function of the time t only, that is,

$$E_0(t)\simeq\sum_{\lambda=1}^{2}\int d^3k\epsilon_x(\vec{k},\lambda)\frac{\sqrt{\hbar\omega/2}}{2\pi}\left[e^{i\theta(\vec{k},\lambda)}e^{-i\omega t}+c.c.\right] , \tag{6}$$

because the large values of $|\vec{k}|$ will not contribute to the motion of a charge bounded by a harmonic force (frequency ω_0). This will be very clear in the next section (see also T.H. Boyer [6]).

In (6), $\theta(\vec{k},\lambda)$ are random phases statistically independent and uniformly distributed in the interval $[0,2\pi]$, $\vec{k}$ is the wave vector such that $|\vec{k}|=\omega/c$, and $\epsilon_x(\vec{k},\lambda)$ is the polarization vector projected in the x axis, with $\lambda=1\,,\,2$. Notice that spectral density of the zero-point radiation is such that [5, 6]

$$\rho_0(\omega)=\frac{\hbar\omega^3}{2\pi^2c^3} . \tag{7}$$

The thermal electric field $E_T(t)$ is also random and is, by assumption, statistically independent from $E_0(t)$. It can be written in a similar manner, namely

$$E_T(t)\simeq\sum_{\lambda=1}^{2}\int d^3k\epsilon_x(\vec{k},\lambda)\frac{h(\omega,T)}{2\pi}[e^{i\theta(\vec{k},\lambda)}e^{-i\omega t}+c.c.] , \tag{8}$$

where T is the absolute temperature and the function $h(\omega,T)$ is given by

$$h(\omega,T)=\sqrt{\frac{\hbar\omega}{2}[\,\coth(\frac{\hbar\omega}{2kT})-1\,]} . \tag{9}$$

Notice that $h(\omega,T)=0$ if $T=0$. The spectral density of the thermal radiation is such that

$$\rho_T(\omega)=\frac{\hbar\omega^3}{\pi^2c^3}\left(\frac{1}{e^{\hbar\omega/kT}-1}\right) . \tag{10}$$

The system we shall study is a charged harmonic oscillator with natural frequency ω_0 and mass m ($mc^2>>\hbar\omega_0$), already considered in a previous work [2]. We shall consider firstly the effects of the zero-point electric

field $E_0(t)$, given in (6), and the radiation reaction force. The electric field associated with the radiation reaction will be denoted by $E_{RR}(t)$ and will be obtained later. The above fields will be introduced into the Schrödinger equation through the vector potential $A_x(t)$ such that

$$-\frac{1}{c}\frac{\partial A_x}{\partial t} = E_0(t) + E_{RR}(t) \ , \tag{11}$$

in the case of *zero-point* radiation, or

$$-\frac{1}{c}\frac{\partial A_x}{\partial t} = E_T(t) + E_{RR}(t) \ , \tag{12}$$

in the case of *thermal* radiation.

2.1) The effects of the zero-point field and the radiation reaction in the Schrödinger equation

For reader convenience we shall obtain, in what follows, an exact solution of the Schrödinger equation by using the same method already presented in the reference [2]. By considering the dipole (or long wavelength) approximation the one dimensional Schrödinger equation takes the form [2]

$$i\hbar\frac{\partial\psi(x,t)}{\partial\, t} = \left[\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} - \frac{e}{c}A_x(t)\right)^2 + \frac{m\omega_0^2 x^2}{2}\right]\psi(x,t) \ , \tag{13}$$

where $A_x(t)$ is x component of the vector potential acting on the charged particle. At this point the exact analytical form of $A_x(t)$ is not known, because the radiation reaction field $E_{RR}(t)$ was not determined. For the moment we shall simply assume that $A_x(t)$ is a c-number that varies with t and is *independent* of x. It should be noticed that this assumption is valid provided that $mc^2 \gg \hbar\omega_0$.

The time independent Schrödinger equation has a ground state solution $\phi_0(x)$ such that

$$\phi_0(x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega_0 x^2}{2\hbar}\right) \ . \tag{14}$$

Moreover, we see that

$$\int_{-\infty}^{\infty} dx\, \phi_0^2(x) x^2 = \frac{\hbar}{2m\omega_0} \ . \tag{15}$$

The time dependent equation (13) has an exact solution that can be written as

$$\psi(x,t) = \phi_0\left(x - q_c(t)\right) \exp\left\{ \frac{i}{\hbar} \left[\left(p_c(t) + \frac{e}{c} A_x(t) \right) x - g(t) \right] \right\} , \qquad (16)$$

where the functions $q_c(t)$, $p_c(t)$ and $g(t)$ are unknown c-numbers that will be determined by the substitution of (16) into (13). This is an old procedure, introduced by Schrödinger (1926) in a famous paper entitled *"The Continuous Transition from Micro to Macro-Mechanics"* (see reference [9], pg. 41). With the above substitution, we get the following equations [10]:

$$p_c(t) = m\dot{q}_c(t) , \qquad (17)$$

and

$$\dot{p}_c(t) = -m\omega_0^2 q_c(t) - \frac{e}{c} \frac{\partial A_x(t)}{\partial t} . \qquad (18)$$

We also obtain the equation $2\dot{g}(t) = \hbar\omega_0 + m\dot{q}_c^2(t) - m\omega_0^2 q_c^2(t)$, which solution can be written as

$$g(t) = \frac{\hbar\omega_0 t}{2} + \frac{m}{2} \int_0^t dt' \left(\dot{q}_c^2(t') - \omega_0^2 q_c^2(t') \right) . \qquad (19)$$

One can combine (17) and (18) to obtain the differential equation

$$m\ddot{q}_c(t) = -m\omega_0^2 q_c(t) + eE_x(t) , \qquad (20)$$

where we have used the fact that $cE_x(t) = -\partial A_x(t)/\partial t$. Notice that, by assumption, every term in (20) is a c-number. According to our definition, the total electric field will be given by

$$E_x(t) = E_0(t) + E_{RR}(t) , \qquad (21)$$

where $E_0(t)$ (see equation (6)) is the classical zero-point field and $E_{RR}(t)$ is the classical radiation reaction field (the particle is *charged*, therefore the radiation reaction field must contribute to $E_x(t)$).

The correct expression for the classical radiation reaction force $eE_{RR}(t)$ is more difficult to obtain because, according to the Schrödinger picture, the charged particle does not have a precise location. One can only say that

$$|\psi(x,t)|^2 = \left(\frac{m\omega_0}{\pi\hbar} \right)^{\frac{1}{2}} \exp\left[-\frac{m\omega_0 (x - q_c(t))^2}{\hbar} \right] , \qquad (22)$$

is the time dependent probability density. Notice that, in order to obtain (22), one must solve (20) which depends on the still undefined radiation

reaction force $eE_{RR}(t)$. This force, however, can be precisely defined in the case of large mass, so that $mc^2 \gg \hbar\omega_0$. In this case, one can safely consider that

$$eE_{RR}(t) \simeq \frac{2e^2}{3c^3} \dddot{q}_c(t) \ , \tag{23}$$

is a good approximation because the Gaussian (22) is so narrow that the harmonically bound particle has a *trajectory.* Based on these considerations we conclude that the expression (23) is valid in the case $mc^2 \gg \hbar\omega_0$, which is consistent with the *long wavelength* approximation. Therefore the equation (20) can be written as

$$\ddot{q}_c(t) + \omega_0^2 q_c(t) \simeq \frac{e}{m} E_0(t) + \frac{2e^2}{3mc^3} \dddot{q}_c(t) \ , \tag{24}$$

where $E_0(t)$ is given by (6). The last term in (24) is responsible for the decay of the excited states of the oscillator.

The stationary solution of the equation (24) is given by [5, 6]

$$q_c(t) = \frac{e}{m} \sum_{\lambda=1}^{2} \int d^3k \epsilon_x(\vec{k},\lambda) \frac{\sqrt{\hbar\omega/2}}{2\pi} \left[\frac{e^{i\theta(\vec{k},\lambda)} e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\frac{2e^2}{3mc^3}\omega^3} + c.c. \right] \ , \tag{25}$$

which is a random real function of the time. This stationary solution is obtained only if $e \neq 0$. If $e = 0$ the equation (24) will lead to oscillatory (non dissipative) coherent states of the harmonic oscillator.

According to the Max Born statistical interpretation of the wave function $\psi(x,t)$, the expectation value of x^2 is given by

$$\overline{x^2(t)} = \int_{-\infty}^{\infty} dx |\psi(x,t)|^2 \, x^2. \tag{26}$$

Taking into account the expressions (16) and (14), one can show that

$$\begin{aligned} \overline{x^2(t)} &= \int_{-\infty}^{\infty} dx \phi_0^2(x - q_c(t)) \ \left[(x - q_c(t))^2 + q_c^2(t) \right] \\ &= \frac{\hbar}{2m\omega_0} + q_c^2(t). \end{aligned} \tag{27}$$

We recall that $q_c^2(t)$ depends on the random phases $\theta(\vec{k},\lambda)$.

From the result (27) we can calculate the mean square value of the particle position. This quantity is obtained by averaging over the random phases present in (25).

The average over the random variables (indicated by the symbol $\langle\,\rangle$) is such that [6]

$$\begin{gathered}\langle e^{i\theta(\vec{k},\lambda)}e^{i\theta(\vec{k}',\lambda')}\rangle = \langle e^{-i\theta(\vec{k},\lambda)}e^{-i\theta(\vec{k}',\lambda')}\rangle = 0\ ,\\ \langle e^{i\theta(\vec{k},\lambda)}e^{-i\theta(\vec{k}',\lambda')}\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\ .\end{gathered} \tag{28}$$

Hence, applying the random average to the expression (27), we obtain

$$\langle\overline{x^2}\rangle = \frac{\hbar}{2m\omega_0} + \langle q_c^2(t)\rangle. \tag{29}$$

Using the stationary solution (25), the average of $q_c^2(t)$ over the random phases is such that [5, 6]

$$\langle q_c^2(t)\rangle = \frac{2e^2}{3\pi m^2c^3}\int_0^\infty d\omega\frac{\hbar\omega^3}{(\omega^2-\omega_0^2)^2+\left(\frac{2e^2}{3mc^3}\right)^2\omega^6}\ . \tag{30}$$

Since $(\frac{2}{3}\frac{e^2}{\hbar c}\frac{\hbar\omega_0}{mc^2})^2 \ll 1$, the integrand of (30) has a very sharp peak at $\omega \approx \omega_0$. Therefore, this integral can be approximated by (see [6] and also the Appendix A of the reference [11])

$$\langle q_c^2(t)\rangle \simeq \frac{2\hbar\omega_0^3e^2}{3\pi m^2c^3}\int_0^\infty \frac{d\omega}{4\omega_0^2(\omega-\omega_0)^2+\left(\frac{2e^2\omega_0^3}{3mc^3}\right)^2}. \tag{31}$$

This expression can be cast in a more simple form, namely

$$\langle q_c^2(t)\rangle = \frac{\hbar\gamma}{4\pi m\omega_0}\int_0^\infty \frac{d\omega}{(\omega-\omega_0)^2+(\gamma/2)^2}, \tag{32}$$

where $\gamma \equiv \frac{2}{3}\frac{e^2\omega_0^2}{mc^3}$, and $\gamma/\omega_0 \ll 1$. This is a standard integral and the result is

$$\langle q_c^2(t)\rangle = \frac{\hbar}{2\pi m\omega_0}[\ \frac{\pi}{2} + \arctan(\frac{2\omega_0}{\gamma})\ ], \tag{33}$$

showing that $\langle q_c^2(t)\rangle$ is charge dependent because $\frac{\gamma}{\omega_0} = \frac{2e^2\omega_0}{3mc^3}$. An expansion of (33) in powers of the small constant γ/ω_0 gives

$$\langle q_c^2(t)\rangle = \frac{\hbar}{2m\omega_0}[\ 1 - \frac{1}{\pi}(\frac{\gamma}{2\omega_0}) + \frac{1}{3\pi}(\frac{\gamma}{2\omega_0})^3 + \cdots\ ]\ . \tag{34}$$

Notice that $\gamma/\omega_0 \approx 10^{-10}$ for an atomic oscillator.

Substituting the result (34) in the expression (29), we get

$$\langle\overline{x^2}\rangle = \frac{\hbar}{m\omega_0}\ , \tag{35}$$

corresponding to a ground state energy that is *twice* the correct value obtained by using the Heisenberg picture. As far we know this discrepancy was first pointed out by A. J. Faria et al. [2].

In the following section we shall show that neither the thermal electromagnetic fields, nor the radiation reaction force, are responsible for this discrepancy. It will be clear that the reason for the discrepancy is that the zero-point fluctuations was considered *twice* in the Schrödinger equation. This was suggested by Faria et al. in the section 5 of their paper [2]. We shall give the detailed proof that their suggestion is correct.

2.2) The effects of the thermal electromagnetic fields and the radiation reaction in the Schrödinger equation

Our first observation refers to the momentum operator used in the Schrödinger equation (13), namely, $p = -i\hbar \frac{\partial}{\partial x}$. This operator already contains the effects of the zero-point electromagnetic field. This can be easily recognized from the works of P. W. Milonni [11, 12]. According to Milonni the commutator between the operators $x(t)$ and $p(t)$ can be calculated within the Heisenberg picture and the result is (see [11], section 2.6)

$$[\ x(t), p(t)\ ] = \frac{ie^2}{m} \frac{8\pi}{3} \int_0^\infty d\omega \frac{\omega\ \rho_0(\omega)}{(\omega^2 - \omega_0^2)^2 + \left(\frac{2e^2}{3mc^3}\omega^3\right)^2} \ , \tag{36}$$

where $\rho_0(\omega)$ is the zero-point spectral density given previously (see our equation (7)). Only $\rho_0(\omega)$ depends on $\hbar$. The calculation of the integral (36) is similar to the calculation of $\langle q_c^2(t) \rangle$ presented within the subsection 2.1. The result is

$$[\ x(t), p(t)\ ] = i\hbar \ . \tag{37}$$

Noticed that the dipole approximation is used in order to obtain the results (36) and (37). The conclusion is that, according to the Heisenberg picture, the constant $\hbar$ appearing in (37) *has its origin in the zero-point radiation with spectral distribution* $\rho_0(\omega) = \hbar\omega^3/2\pi^2c^3$. As we said above this was shown by P.W. Milonni in the references [11, 12]. In words of Faria et.al. [2]: "the kinetic energy operator used in the Schrödinger picture, namely, $\frac{p^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$, is the natural channel by means of which the zero-point electromagnetic fluctuations are incorporated into the Schrödinger equation for a charged particle".

Considering these observations, we conclude that *only* the radiation reaction and the thermal electromagnetic fields of the vacuum can be consistently introduced within the Schrödinger equation (13). However, it is possible to

replace the zero-point field $E_0(t)$, used in the equation (21), by the thermal random field $E_T(t)$ given by (8). We shall show that this replacement will lead to correct results for several observable quantities. The discussion of the effects of the thermal fields is an interesting and clarifying example.

Notice that the total electric field acting on the charged particle will be $E_x(t) = E_T(t) + E_{RR}(t)$, where $E_{RR}(t)$ is the radiation reaction field (see (23)). As before the vector potential $A_x(t)$ is such that $E_x(t) = -\frac{1}{c}\frac{\partial A_x}{\partial t}$.

Following the calculations explained within the subsection 2.1 (see the equations (16) to (24), it is straightforward to show that the function $q_c(t)$ will be given by the equation (25), with the replacement of $\sqrt{\hbar\omega/2}$ by $h(\omega, T)$ introduced in (9). Moreover, it is easy to show that $\langle q_c^2(t)\rangle$ will be given by

$$\langle q_c^2(t)\rangle = \frac{2e^2}{3\pi m^2 c^3}\int_0^\infty d\omega \frac{\hbar\omega^3\ [\ \coth(\frac{\hbar\omega}{2kT}) - 1\ ]}{(\omega^2-\omega_0^2)^2 + \left(\frac{2e^2}{3mc^3}\right)^2\omega^6}, \tag{38}$$

instead of our previous equation (30). Notice that $\coth(\frac{\hbar\omega}{2kT}) - 1 = (e^{\frac{\hbar\omega}{kT}} - 1)^{-1}$.

Introducing again the constant $\frac{\gamma}{\omega_0} = \frac{2}{3}\frac{e^2}{\hbar c}\frac{\hbar\omega_0}{mc^2} \ll 1$, the above integral can be calculated with the same approximations used previously (subsection 2.1). With this procedure we get

$$\begin{aligned} \langle q_c^2(t)\rangle &= \frac{\hbar}{m\omega_0}\left(\frac{1}{e^{\hbar\omega_0/kT}-1}\right)\left[1 - \frac{1}{\pi}\left(\frac{\gamma}{2\omega_0}\right) + \frac{1}{3\pi}\left(\frac{\gamma}{2\omega_0}\right)^3 + \cdots\right] \\ &\simeq \frac{\hbar}{m\omega_0}\left(\frac{1}{e^{\hbar\omega_0/kT}-1}\right). \end{aligned} \tag{39}$$

This new result combined with our previously expression (29) gives

$$\langle\overline{x^2}\rangle = \frac{\hbar}{2m\omega_0}\left(1 + \frac{2}{e^{\hbar\omega_0/kT}-1}\right). \tag{40}$$

This is the *correct* value of $\langle\overline{x^2}\rangle$ for an *arbitrary* temperature T. Notice that we get $\langle\overline{x^2}\rangle = \frac{\hbar}{2m\omega_0}$ when $T = 0$. This is expected on physical grounds, and is in agreement with calculation based on the Heisenberg picture.

The conclusion is that the discrepancy between the Heisenberg and the Schrödinger pictures, pointed out by A.J. Faria et al. [2], is eliminated *only* when we *remove* the zero-point field $E_0(t)$ from the Schrödinger equation (13). *The effects of the zero-point field are already included in the operator* $-i\hbar\partial/\partial x$. The inclusion of the fluctuating thermal electromagnetic fields into the Schrödinger equation is necessary and helps in the clarification of this point.

We also would like to present another interesting effect of the inclusion of the thermal electromagnetic fields (see (8) and (9)) into the Schrödinger equation (13). We recall that (see (22))

$$|\psi(x,t)|^2 = \left(\frac{m\omega_0}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left[-\frac{m\omega_0}{\hbar}(x-q_c(t))^2\right] , \tag{41}$$

where the fluctuating coordinate $q_c(t)$ depends on the temperature T, and is such that

$$q_c(t) = \frac{e}{m}\sum_{\lambda=1}^{2}\int d^3k\ \epsilon_x(\vec{k},\lambda)\frac{h(\omega,T)}{2\pi}\left[\frac{e^{i\theta(\vec{k},\lambda)}e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega}+c.c.\right] . \tag{42}$$

A quantity of interest is the temperature dependent probability density

$$P_T(x) \equiv \langle |\psi(x,t)|^2\rangle , \tag{43}$$

where the symbol $\langle\ \rangle$ indicates average over the random phases $\theta(\vec{k},\lambda)$.

In order to calculate (43) we shall introduce the Fourier transform

$$\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{-\frac{\hbar k^2}{4m\omega_0}}\ e^{-ik(x-q_c)} = \sqrt{\frac{m\omega_0}{\pi\hbar}}\ \exp[-\frac{m\omega_0}{\hbar}(x-q_c(t))^2]. \tag{44}$$

With this notation we obtain

$$P_T(x) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{-\frac{\hbar k^2}{4m\omega_0}}\ e^{-ikx}\langle e^{ikq_c(t)}\rangle. \tag{45}$$

The calculation of the characteristic function $\langle e^{ikq_c(t)}\rangle$ is standard. One can show that

$$\langle e^{ikq_c(t)}\rangle = \sum_{n=0}^{\infty}\frac{(ikq_c(t))^n}{n!} = 1 - \frac{k^2\langle q_c(t)^2\rangle}{2!} + \frac{k^4\langle q_c(t)^4\rangle}{4!} + \cdots , \tag{46}$$

or

$$\langle e^{ikq_c(t)}\rangle = \sum_{n=0}^{\infty}(-1)^n\ \frac{k^{2n}\langle q_c(t)^{2n}\rangle}{(2n)!} . \tag{47}$$

According to Boyer, (see eq. (68) of reference [6]), we have

$$\langle q_c(t)^{2n}\rangle = \frac{(2n)!}{n!\ 2^n}\left(\frac{\hbar[\coth(\frac{\hbar\omega_0}{2kT})-1]}{2m\omega_0}\right)^n , \tag{48}$$

where the factor $\left[\coth(\frac{\hbar\omega_0}{2kT})-1\right]$ has its origin in the expressions (42), for $q_c(t)$, and (9) for $h(\omega,T)$. Using (48) and (47) into (45) we obtain

$$\langle|\psi(x,t)|^2\rangle = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{-ikx}\exp\left[-\frac{\hbar k^2}{4m\omega_0}\coth(\frac{\hbar\omega_0}{2kT})\right]. \tag{49}$$

The integration is straightforward leading to the result

$$P_T(x) = \langle|\psi(x,t)|^2\rangle = \sqrt{\frac{m\omega_0}{\pi\hbar\coth(\frac{\hbar\omega}{2kT})}}\exp\left[-\frac{m\omega_0 x^2}{\hbar\coth(\frac{\hbar\omega_0}{2kT})}\right], \tag{50}$$

valid for an *arbitrary* temperature T. This result is observed experimentally.

Moreover, this expression for the quantum probability distribution $P_T(x)$, associated with the probabilistic motion of a charged harmonic oscillator, was derived here in a simple and direct manner. It coincides with the result presented by R.W. Davies and K.T.R. Davies [13]. These authors have used the Wigner phase space distribution functions associated with the discrete excited states of the harmonic oscillator, obtained according to the Schrödinger picture. In their work, the temperature effects are introduced by the use of the Boltzmann factors associated with each excited state. Notice that Davies and Davies [13] do not mention, neither the effects of the thermal electromagnetic fields (see (8)), nor the dynamical role of the radiation reaction force. In this respect our calculation elucidates in details, and more clearly, the influence of the radiation reaction and the radiation bath on the oscillator.

3. Conclusions

The SED approach is mainly used in the study of linear systems or other systems that can be treated linearly in a good approximation. Interesting examples are the interaction of electric and magnetic dipoles with simple circuits with thermal and zero-point voltage fluctuations, as the RLC circuit. New findings were obtained in this way, and were published recently [14, 15, 16, 17, 18]. We call the reader attention to the work of Blanco et.al. [17] concerning the enhancement of the voltage fluctuations by the action of the classical zero-point magnetic field in the coils of an appropriated constructed solenoid. This new prediction of SED is currently under experimental investigation by L.J. Nickisch [19]. A nonlinear phenomenon, namely, the "tunneling" from a potential well with a barrier, was successfully explained as an effect of the zero-point radiation [20].

In our paper we have studied a fundamental problem which is possible to be treated within the realm of SED. We have extended the analysis of A.

J. Faria et al. [2], by considering the effects of the *zero-point* and *thermal* electromagnetic fields in a harmonic oscillator using the *Schrödinger picture*. The effects of the radiation reaction were also correctly taken into account. We concluded that *the effects of the zero-point radiation are already included into the Schrödinger equation (13) by means of the momentum operator* $p = -i\hbar\frac{\partial}{\partial x}$ (see also the references [2, 3, 12] which complement this statement). In our opinion, this is the most important finding of our work. Such a conclusion stresses the problem of the *equivalence* between the Schrödinger and the Heisenberg pictures of Quantum Mechanics (see the references [1, 2]) as far the harmonic oscillator is concerned. This is easy to understand because the effects of the zero-point field (6) has to be *subtracted*, in the Schrödinger treatment of the oscillator, in order to obtain the correct result (see the eq. (35) in the subsection 2.1). This *subtraction is not necessary in the case of the Heisenberg picture* [2, 11].

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